Thinking about Alternative Procedures and Algorithms for Computation

(Excerpt from *"Building Support for School Mathematics: Handbook for Working with Parents and the Public by R. Parker and J. Akers-Mitchell,* Heinemann, 2006)

Some parents report that they have been told that their child's teacher does not care about algorithms. If true, this is unfortunate. It is likely that this miscommunication is the result of confusion about the term algorithm. Although we are all familiar with the standard paper and pencil algorithms for addition, subtraction, multiplication and other arithmetic operations, there does seem to be a fair amount of confusion about what an algorithm is. Simply put, an algorithm is a set of rules for solving a particular kind of problem. Algorithmic thinking is central to the doing of mathematics. Many of us were taught that there is one best algorithm for each type of computation. In reality though, there are a variety of algorithms, some of which work more efficiently than the standard paper and pencil algorithms when it comes to reasoning about and with numbers.

It is important to understand that when children are asked to reason with numbers in their own way, not every procedure that a child invents is efficient. Nor are all of their invented procedures necessarily algorithms. Given frequent opportunities to reason with number, children gravitate to the most efficient procedures that work consistently over time. A useful resource on the difference between procedures and algorithms is an illuminating article written by Hyman Bass, professor of mathematics at Michigan State University ("Computational Fluency, Algorithms, and Mathematical Proficiency: One Mathematician's Perspective" by Hyman Bass in *Teaching Children Mathematics*, February, 2003, Volume 9, Issue 6).

It is not uncommon to hear parents ask a question such as, *"My child comes home with very strange ways of adding and subtracting. It doesn't look anything like the way I learned. I'm not sure why this is happening, and I don't know how to help her. What can I do?"* This question gets to the heart of one of the big changes taking place in some mathematics classrooms. Rather than memorize the standard computation procedures most of us learned in school, in some of the newer programs, children are asked to invent and/or try diverse procedures. In doing so, they are encouraged to make sense of what they are doing and explain their reasoning.

Many of us were taught that there is one right algorithm, or procedure, for solving each type of computation problem. Yet, some other countries use different algorithms than those we use. These diverse algorithms work perfectly well. However, when children from these countries come here, many teachers believe that their job is to teach these children to abandon the procedures they were taught, even though the algorithms are effective and efficient, and learn to compute 'the right way'.

There are two main concerns about the limitations of teaching standard paper and pencil algorithms, and especially about teaching them prematurely. First, the algorithms often obscure place value relationships, and second, they can interfere with the development of children's ability to reason with numbers.

Addition

There is a great deal of evidence that children in this country lack a fundamental understanding of place value that is essential to numerical reasoning. To examine the role that the teaching of paper and pencil algorithms may play in undermining children's understanding of place value, consider the following example. When learning to solve the problem, children are told to start on the right.

$$\begin{array}{r} {}^{1\,1} \\ 178 \\ \underline{+\,546} \\ 724 \end{array}$$

Add 8 and 6, the sum is 14. Write down the 4 and carry the 1 (although it is not really a 1 that is carried, it is a 10).
Add the 7 and 4, then add the 1 you carried. The sum is 12. Write down the 2 and carry the 1 (although it is not really a 1, but 100. Also, the 7 and 4 that are added are not really 7 and 4, they are 70 and 40).

When we teach children these procedures mechanically, we actually teach things about numbers that are not true rather than helping them understand the actual place value relationships involved. Another concern is that nothing in the procedure above, moving right to left, is useful in determining the reasonableness of the answer.

Now let us look at a procedure many children use naturally when we do not first teach them the method above. Most often they add from left to right, the same way they read. When children solve multi-digit addition problems they often start on the left and say,

$$\begin{array}{r} 178 \\ \underline{+\,546} \\ 724 \end{array}$$

100 and 500 is 600. (Note that they can immediately tell what a reasonable answer will be, and they are using the real relationships in the numbers.)
70 and 40 is 110, and add that to the 600 to get 710.
8 and 6 is 14, and when added to 710, the total is 724.

Because children are using number relationships that make sense to them, they are more confident in their work and can defend why their answers make sense.

Subtraction

Let us look at the following subtraction problem. When children are taught to use the standard paper and pencil algorithm, they are told,

$$\begin{array}{r} {}^{5}\phantom{{}^{1}3} \\ \cancel{6}\,{}^{1}3 \\ -\,\underline{2\;7} \end{array}$$

You cannot take 7 from 3, so borrow 1 from the 6, and make it a 5. Put the 1 by the 3, making it a 13.
Now subtract 7 from 13 to get 6. Then subtract 2 from 5 to get 3.

3 6

Once again, notice the misconceptions about place value that are being taught in the process. The number is not really 6 it is 60. The 6 does not become 5, it is changed to 50. It is not a 1 that is put by the 3, but, rather, it is a 10, etc.

Unfortunately, because this is an abstract recipe for subtraction that confuses many children, subtraction gets re-taught year after year, beginning in first grade, and often continuing through 6^{th} grade. If children learn this specific algorithm correctly, it works consistently and efficiently when doing subtraction problems with paper and pencil. However, it does not help children learn to judge the reasonableness of their answer, and the algorithm is not efficient when it comes to calculating mentally.

Now let us consider some ways that children naturally subtract when they are not first taught the algorithm above. Again, children tend to begin on the left. This is one common approach they use:

$$\begin{array}{r} 63 \\ -\underline{27} \end{array}$$

Subtracting 20 from 63 is 43.
I can break the 7 in the 27 into 3 and 4, and taking 3 from 43 is 40.
Then taking 4 from 40, the answer is 36.

Other children subtract 30 from 63 to get 33, then add back the three extra they took away for an answer of 36.

Some children prefer addition to subtraction and are happy to learn that it is very easy to add to get an answer to any subtraction problem.

$$\begin{array}{r} 63 \\ -\underline{27} \end{array}$$

Add up by beginning with 27 and add 3 to get to 30.
Add 33 to 30 to get to 63.
Since I added 3 and 33 or 36, the difference between 27 and 63 is 36.

Some children add three to both numbers in the problem above so that they can take away a multiple of 10 since that is very easy to do.

$$\begin{array}{r} 63 \\ \underline{-27} \end{array} \rightarrow \begin{array}{r} 66 \\ \underline{-30} \end{array}$$

Add 3 to both 63 and 27.
66 minus 30 is 36.

An algorithm that some children gravitate to involves negative numbers. They again start from the left

$$\begin{array}{r} 63 \\ -\underline{27} \end{array}$$

20 from 60 is 40.
7 from 3 is negative 4.
40 + $^{-}4$ is 36.

Notice how much more children are learning about number relationships when they have opportunities to make sense of numbers in their own ways and when they get to see a variety of different ways to solve computation problems.

Another major concern about the impact of teaching standard paper and pencil algorithms is that this practice typically results in children and adults who are unable to reason effectively with numbers. In fact, although we have spent a great deal of math instructional time teaching these algorithms, there is a large body of research that suggests that children continue to get wrong answers to computation problems. Worse yet, they don't even notice they are wrong since they have no way of determining the reasonableness of their answers.

Multiplication

Looking at how children naturally solve a multiplication problem helps show how this approach will better prepare children for future success with mathematics. In doing so, we will explore the relationship between the natural procedures that children use and their preparation for algebra. Specifically we will look at the commutative, associative and distributive properties, three very important mathematical ideas that students encounter, and many struggle with, during algebra courses.

This is the way most of us were taught to do multi-digit multiplication

4	
2 6	First multiply 8 times 6. The product is 48. Write down the 8 and carry the 4.
x 1 8	Then multiply 8 times 2. The product is 16. Add the 4 you carried
2 0 8	for a total of 20. Write the 20 in front of the 8.
2 6 0	Next, bring down a zero and write it under the 8 in 208.
4 6 8	Then multiply 1 times 6, etc.

Notice that this procedure once again, does not focus on the real relationships in the numbers. Nor does it give children the tools to determine the reasonableness of their answers.

When children are encouraged to make sense of multiplication problems in their own ways, they again use a variety of methods to do so. Their invented methods require children to use numerical reasoning, and some of their methods are much more efficient than the standard paper and pencil algorithm.

Many children use what is known as partial products to solve problems like these.

2 6	
x 1 8	
2 6 0	Multiply 10 times 26 for a partial product of 260.
1 6 0	Then multiply 8 times 20 for a second partial product of 160.
4 8	Multiply 8 times 6 to get 48 for the third partial product.
4 6 8	Add the partial products: 260 and 160 is 420, and 48 more is 468.

Some children do the following: Add 2 to 18 to make 20 since it is easy to multiply by 20.
Multiply: 20 x 26 = 520
Subtract two 26s or 52 from 520 by breaking 52 into 20 + 32.
(520 – 20 = 500, 500 – 32 = 468)

Some children find it easier to change the problem as follows:

$$\begin{array}{r} 26 \\ \underline{\times 18} \end{array} \rightarrow \begin{array}{r} 52 \\ \underline{\times \;\; 9} \end{array}$$

Double one number and halve the other.
(26 x 2 = 52) and ($\frac{1}{2}$ of 18 = 9)
Then, 9 x 50 = 450 and 9 x 2 = 18, so 450 + 18 = 468.

When we look at children's invented procedures for multiplication problems, we find that not only do they use the number relationships, and have confidence in their procedures, they also naturally use the commutative, associative and distributive properties of Mathematics.

With multiplication of whole numbers, for example, the commutative property states that *a* x *b* = *b* x *a*. Children who have learned to reason with numbers understand this. They know that they can see the above problem as 18 x 26 or as 26 x 18, whichever gives them easier access into the problem.

Likewise, the distributive property states that *a(b* + *c)* = *ab* + *ac*. Again, children who have learned to reason with numbers understand this at an early age. They know that they can think about the problem above as
(26 x 10) + (26 x 8). Or they can break the 26 apart into (25 + 1) and see the problem as (18 x 25) + (18 x 1). Multiplying by 25 is easy for students who know that every four 25s is 100, so 18 x 25 is 450. Adding 18 more makes the product 468. Others think of multiplying by 25 as taking 1/4 of 100 times the number, in this case, 1/4 of 1800 which is 450. The students know that there are several ways to take the numbers apart and redistribute them in order to create a problem that is easier to solve.

The associative property states that *(a* x *b) c* = *a (b* x *c).* Once again, children learn to use this property at an early age when they learn to make sense of different procedures for multiplication and addition. In the problem above, the child who changes the problem 18 x 26 to 9 x 52, is using the associative property or (9 x 2) 26 = 9 (2 x 26) as well as other mathematical ideas such as factoring.

The arithmetic properties historically have been a stumbling block for many students who are taking algebra courses. Trying to learn the concepts in an abstract manner is often confusing. On the other hand, students who have learned to reason with numbers have been using the underlying mathematics of those properties for years before getting to algebra. All these students need to do, then, is learn an abstract way to label mathematical relationships they already understand—a far easier task.

Division

Does it still make sense to teach long division in this technological age? Teaching children the standard paper and pencil algorithm consumes a great deal of mathematics instruction time—typically, beginning in 3rd grade and continuing through the 8th grade. Still, high school teachers complain that students' answers to multiplication and division problems are frequently off by ten or even 100 fold. To make matters worse, students rarely notice that their answers are so far off since they are simply following procedures they have memorized rather than making sense of the problem. We also need to consider that technology, for the most part, has replaced the need to do long division with paper and pencil. No jobs today require workers to do long division problems with paper and pencil. In the workplace of today, division problems are done exclusively by machines.

The amount of instructional time spent teaching the standard paper and pencil division algorithm far outweighs the value of teaching the algorithm to mastery. Still, it is important for children to understand division, to know when to use it, and to be able to judge the reasonableness of the results of a long division problem. There are better ways of teaching long division—ways that once again help children understand the relationships involved. In order to illustrate this, we will look at three different division problems.

The following was the first division problem given to a class of 5th graders before they were introduced to the standard long division algorithm.

$$130 \div 5$$

The teacher was interested in finding out what children would naturally do with division prior to instruction. The children were asked to reason through the problem mentally and share their thinking. This is what they said:

Child A: There are 6 fives in 30, and 20 fives in 100, so there are 26 fives in 130.

Child B: I did the opposite. There are 20 fives in 100, and 6 fives in 30 for a total of 26 fives in 130.

Child C: I know there are 2 fives in 13, and 6 fives in 30, so I got 26.

When Child C was asked why she took 5 into 30, she stumbled several times in her explanation then replied, "Oh never mind." Although she was correctly applying a shortcut algorithm she had been taught at home, when asked to explain her reasoning, she was unable to because she did not understand the relationships involved. When asked a question, she abandoned the idea rather than defend it.

Child D: I divided 130 by 10 and got 13. Then I doubled the 13, because I was only supposed to divide by half as much or 5.

It was interesting to see how these children naturally approached division before instruction.

Let us look at how these same children were approaching division four weeks later. Again, these fifth graders were asked to solve the problem mentally. Before reading on, you may want to think about how you might do this.

$$578 \div 23$$

	578	CHILD A
20 x 23	<u>460</u>	20 twenty-threes equals 460, which leaves 118 from the 578.
	118	If 10 twenty-threes are 230, then 5 twenty-threes would be half of that or 115.
<u>5</u> x 23	<u>115</u>	I took 115 from the 118 that was left. This left me with a remainder of 3 out of 23.
25	3	Altogether I took away 25 groups of 23 for an answer of $25\frac{3}{23}$

	578	CHILD B
10 x 23	<u>230</u>	I started by taking away just 10 groups of 23 or 230.
	348	That left me with 348.
10 x 23	<u>230</u>	Then I took another 10 groups of 23 away, which left me with 118.
	118	
<u>5</u> x 23	<u>115</u>	Then I took away 5 groups of 23, or 115. I was left with a remainder of 3.
25	3	

Let us look at one more problem in order to examine another approach that these 5th graders learned to use for long division problems.

$$604 \div 14$$

14		One child approached the problem by first listing multiples of 14.
28		
42		
56 X 10 = 560		She then multiplied the 4th multiple by 10 (14 x 4 x 10).
14)604		
<u>560</u>	40	She subtracted 40 groups of 14 or 560 and was left with 44.
44		
<u>42</u>	<u>3</u>	She looked at her list of multiples, subtracted 3 more 14s or 42, and was left with 2.
2	43	She subtracted 40 + 3 groups of 14. Her answer is $43\frac{2}{14}$, or $43\frac{1}{7}$.

Another child did it this way.

10 x 14 = 140,	10 x 14 is 140,
20 x 14 = 280	so 20 times 14 would be 280.

604		
– 280	20 14s	280 from 604 is 324.
324		
– 280	20 14s	Subtracting another twenty 14s or 280 leaves 44.
44		
– 28	2 14s	Subtracting two more 14s or 28 is 16.
16		
– 14	1 14	Subtracting one more 14 gives an answer of 43 remainder 2, or $43\frac{2}{14}$, or $43\frac{1}{7}$, depending on the context.
2	43	

Yet another child began by dividing both the numerator and denominator by 2 in order to change the problem 604 ÷ 14 to 302 ÷ 7.

7	He listed multiples of 7
14	
21	
28 X 10 = 280	He knew that 280 = 7 X 4 X 10 = 7 X 40
35	

$7\overline{)302}$		
– 280	40 groups of 7	He took 40 groups of 7, or 280, from 302 leaving him with 22.
22		
– 21	3 groups of 7	He then took 3 groups of 7, or 21 away, leaving him with an answer of $43\frac{1}{7}$.
1		

Consider for a moment the understanding of number these children are demonstrating as they use and explain their alternative methods of solving long division problems. The methods these children have learned to use naturally can be very helpful when it comes to estimating or determining the reasonableness of an answer to a long division problem.

Summary

When children are given frequent opportunities to reason with numbers while solving problems, it is often surprising what they are able to do. Many teachers and parents comment that they are amazed to see the sophisticated thinking children use and how they are able to solve problems with relative ease.

When children are taught standard paper and pencil algorithms that they do not understand, they often come to believe that math is about memorizing recipes that don't make sense. Many end up feeling that they are not any good at math. On the other hand, children who have been taught to reason with numbers know that mathematics is not about memorizing recipes. It is about understanding relationships. They know they can make sense of mathematics, and they are confident in explaining how they know their answers make sense.

What can parents do to help? They can play with numbers with their children whenever possible—when riding in the car or even at the dinner table; explore new ways of doing addition, subtraction, multiplication and division problems mentally; explain to their child how they know their approach to a problem makes sense, and listen to their child explain his or her thinking. They can make this a fun time as a family, and model being a willing problem solver. This sends the important message that mathematics is both engaging and useful.

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